

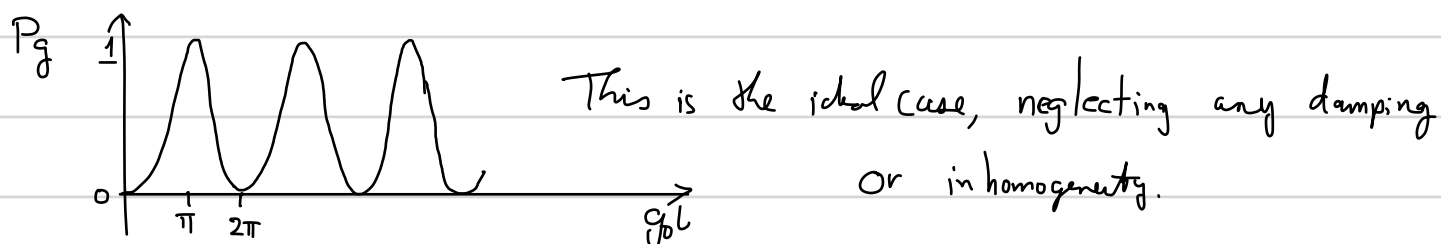
Physics 566 - Quantum Optics I
Problem Set 8 - Solutions

Problem 1: Cavity QED

(a) Each atom is prepared in $|e\rangle$ is sent through the cavity in vacuum. The joint atom-cavity system evolves according to the Jaynes-Cummings Hamiltonian. Since we begin in the 1-excitation subspace,

$$|\Psi_{AF}(t)\rangle = \cos(g_0 t) |e, 0\rangle + i \sin(g_0 t) |g, 1\rangle$$

The probability of finding the atom in the ground state is $\sin^2(g_0 t)$ - This is vacuum Rabi flopping



Note: When $g_0 t = \pi$, we deterministically deposit one photon in the cavity.

(b) We now send pairs of atoms through the cavity. The first is prepared in $|e\rangle$ and after at time $2g_0 = \pi$, $|\Psi_{AF}\rangle = i |g\rangle \otimes |1\rangle$. After a time T the second atom sent through the cavity. Because the photon has a finite lifetime in the cavity, the probability amplitude to have one photon in the cavity decays, by the effective Hamiltonian

$$\Rightarrow |1\rangle_F \rightarrow e^{-\frac{\Gamma_c T}{2}} |1\rangle_F = e^{-\frac{\Gamma_c T}{2}} |1\rangle_F = e^{-\frac{T}{2\tau_c}} |1\rangle_F, \text{ where } \tau_c = \text{lifetime}$$

If we condition on measuring the first atom in $|g\rangle$, then we know, right at that time, we have one photon in the cavity. After time T , the cavity decays back to the vacuum, and a second atom, now prepared in $|g\rangle$ is sent through the cavity for another π -pulse.

The resulting joint evolution of atom-2 + field $\Rightarrow e^{-T/2\tau_c} |e\rangle_2 \otimes |0\rangle$

$$\Rightarrow \underbrace{P_{e_2|g_1}}_{\uparrow} = |e^{-T/2\tau_c}|^2 = e^{-T/\tau_c}$$

Conditional probability of measuring atom-2 in $|e\rangle$ having measured atom-1 in $|g\rangle$.

(c) We use the Ramsey interferometer to measure coherence in the atom. Here we study the transfer of quantum coherence between two atoms mediated by a cavity.

The evolution of the first atom + cavity:

$$|e\rangle_1 \otimes |0\rangle \xrightarrow{R_1} \left(\frac{|e\rangle_1 + i|g\rangle_1}{\sqrt{2}} \right) \otimes |0\rangle \xrightarrow{C} |g\rangle_1 \otimes \left(\frac{|1\rangle + i|0\rangle}{\sqrt{2}} \right)$$

(Note, the Jaynes-Cummings π -pulse swapped the qubit between the atom + cavity.)

Now we send the second atom through the cavity after time T . Ignoring, for the moment the cavity decay, the ideal evolution is

$$|g\rangle_2 \otimes \left[\frac{|1\rangle + i|0\rangle}{\sqrt{2}} \right] \xrightarrow{C} \frac{|g\rangle_2 \otimes |1\rangle + i|g\rangle_2 \otimes |0\rangle}{\sqrt{2}} \xrightarrow{C} \left(\frac{|e\rangle_2 + i|g\rangle_2}{\sqrt{2}} \right) \otimes |0\rangle$$

We have thus swapped the coherence from atom-1 to the cavity and then from the cavity into atom-2.

The second Ramsey pulse now measures the coherence in atom 2. From lecture 7 we showed that the second $\pi/2$ pulse is phase-shifted relative to the first, the Ramsey fringe will be

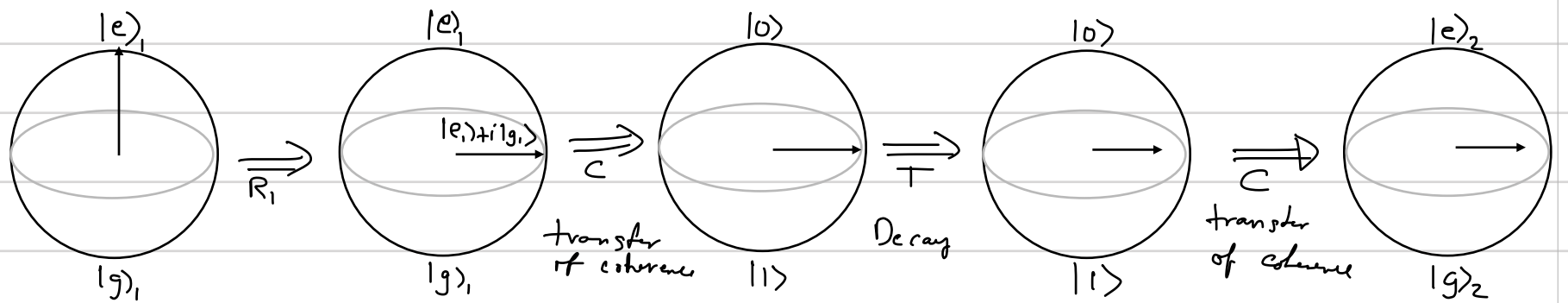
$$P_{e_2|g_1} = \frac{1}{2} (1 + \cos\phi e^{-T/T_2}) \quad \text{where } T_2 = \text{coherence decay time}$$

Here $T_2 = \frac{T_c}{2} = \frac{1}{2}$ lifetime in cavity since according to the effective Hamiltonian

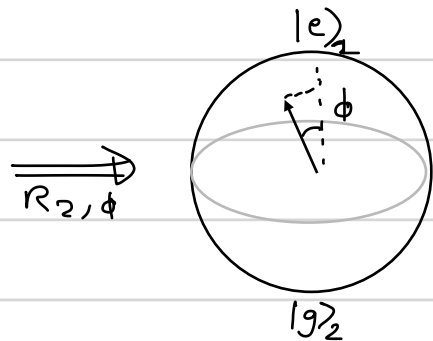
$$\frac{|1\rangle + i|0\rangle}{\sqrt{2}} \xrightarrow{e^{-T/2T_c}} \frac{e^{-T/2T_c} (|1\rangle + i|0\rangle)}{\sqrt{2}} \quad \text{for the cavity coherence, which is then transferred to the atom.}$$

$$\Rightarrow P_{e_2|g_1} = \frac{1}{2} (1 + \cos\phi e^{-T/2T_c})$$

Note, in contrast to part (a), the decay is half as fast because the Ramsey fringe depends on T_2 , where as the probability of decay in (a) depends on T_1 : $T_2 = \frac{1}{2}T_1$ for lifetime decay.



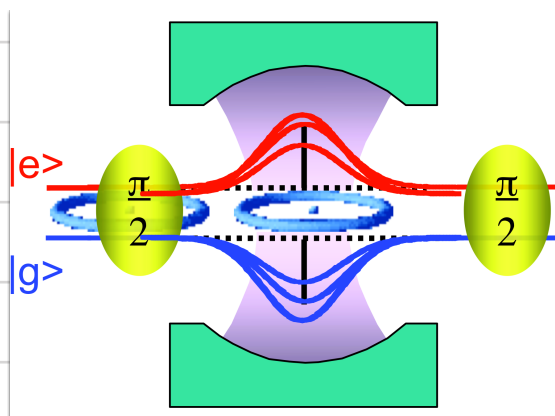
Read out coherence
in atom-2



For a review of these experiments by
the Haroche group see: S. Haroche, M. Brune,
and J.M. Raimond, Phil Trans R. Lond A.
355, 2367 (1997).

(d) We now consider an off-resonant interaction between the atom and cavity mode. In this circumstance, there will not be emission or absorption of "real photons" into the cavity, but instead a kind of "virtual emission and absorption." This leads to a light shift, or joint phase shift on the atom/cavity system. The amount of light shift will depend on the number of photons in the cavity, so this apparatus allows us to measure the # of photons in the cavity without absorbing any photons \Rightarrow Quantum Nondemolition (QND) measurement.

The evolution is assumed to be adiabatic \Rightarrow The system follows the dressed eigenstates as sketched in the figure below (credit to Prof. Haroche).

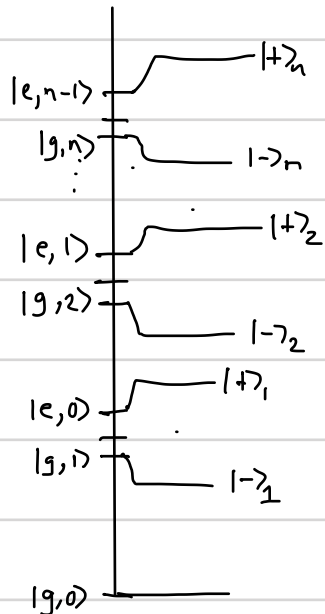


(Slow for red detuning)

Recall the dressed energy levels with N_T -excitations $E_{\pm}(N_T) = \hbar\omega_c(N_T - \frac{1}{2}) \pm \frac{\hbar}{2} \sqrt{\Delta^2 + N_T(2g)^2}$

Dressed States: $|\pm_{N_T}\rangle = \cos\frac{\Theta_{N_T}}{2} |g, N_T\rangle \pm \sin\frac{\Theta_{N_T}}{2} |e, N_T\rangle$; $\tan\Theta_{N_T} = \frac{-\sqrt{N_T} 2g}{\Delta}$

Consider red detuning. The Jaynes-Cummings Ladder



The Bare state $|g, n\rangle$ adiabatically connects to $|-\rangle_n$
The Bare state $|e, n\rangle$ adiabatically connects to $|+\rangle_{n+1}$

The energy shift of these levels leads to a phase shift:

$$\Delta E_{g,n} = E_{gn}^{\text{dressed}} - E_{gn}^{\text{bare}}, \quad \Delta E_{e,n} = E_{en}^{\text{dressed}} - E_{en}^{\text{bare}}$$

$$E_{gn}^{\text{bare}} = -\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c \Rightarrow \Delta E_{gn} = -\frac{\hbar\Delta}{2} - \frac{\hbar}{2}\sqrt{\Delta^2 + n(2g)^2}$$

$$E_{en}^{\text{bare}} = +\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c \Rightarrow \Delta E_{en} = +\frac{\hbar\Delta}{2} + \frac{\hbar}{2}\sqrt{\Delta^2 + (n+1)(2g)^2}$$

Apply the Ramsey $\frac{\pi}{2}$ -pulse to the atom, and then passing it through the cavity with $|n\rangle$ photons, the evolution of atom-cavity system:

$$|e\rangle \otimes |n\rangle \Rightarrow \left(\frac{|e\rangle + i|g\rangle}{\sqrt{2}} \right) \otimes |n\rangle = \frac{|e, n\rangle + i|g, n\rangle}{\sqrt{2}} \Rightarrow \frac{e^{-i\delta_{e,n}} |e, n\rangle + i e^{-i\delta_{g,n}} |g, n\rangle}{\sqrt{2}}$$

$$\text{where } \delta_{e,n} = \frac{\Delta E_{en} t}{\hbar} = \left(\frac{\hbar}{2\hbar} \sqrt{\Delta^2 + (n+1)(2g)^2} + \frac{\Delta}{2} \right) t$$

$$\delta_{g,n} = \frac{\Delta E_{gn} t}{\hbar} = \left(\frac{\hbar}{2\hbar} \sqrt{\Delta^2 + n(2g)^2} - \frac{\Delta}{2} \right) t$$

$$\equiv \left(\frac{|e\rangle + i e^{i\Phi_n} |g\rangle}{\sqrt{2}} \right) \otimes |n\rangle \quad (\text{up to an overall phase})$$

where $\Phi_n = \delta_{e,n} - \delta_{g,n} =$ Rotation angle of atom in Bloch sphere.

\Rightarrow We can measure Φ_n in the Ramsey interferometer.

By applying the second $\frac{\pi}{2}$ -pulse in R_2 , with relative phase ϕ , the Ramsey fringe

$$P_e(\phi) = \cos^2(\phi - \Phi_n)$$

The photon # is then encoded in the phase of the Ramsey fringe!

See Gleyzes et al., Nature 446, 297 (2007).

Problem 2: The relative role of vacuum fluctuations + radiation reaction in spontaneous emission

(a) The Hamiltonian for the two-level atom interacting with the quantized field in the RWA:

$$\hat{H} = \frac{\hbar\omega_0}{2} \hat{\sigma}_z + \sum_{\vec{k}, \mu} \hbar\omega_k \hat{a}_{\vec{k}, \mu}^\dagger \hat{a}_{\vec{k}, \mu} + \sum_{\vec{k}, \mu} \hbar (g_{\vec{k}, \mu} \hat{\sigma}_+ \hat{a}_{\vec{k}, \mu} + g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_-)$$

Note: Here, in the Heisenberg picture, all operators are evaluated at the same time and commute.

Heisenberg equations of motion: (using $[\hat{a}_{\vec{k}, \mu}, \hat{a}_{\vec{k}', \mu'}^\dagger] = \delta_{\vec{k}, \vec{k}'} \delta_{\mu, \mu'}$, $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$, $[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$)

$$\frac{d}{dt} \hat{a}_{\vec{k}, \mu} = -\frac{i}{\hbar} [\hat{a}_{\vec{k}, \mu}, \hat{H}] = -i\omega_k \hat{a}_{\vec{k}, \mu} - i g_{\vec{k}, \mu}^* \hat{\sigma}_-$$

$$\begin{aligned} \frac{d}{dt} \hat{\sigma}_+ &= -\frac{i}{\hbar} [\hat{\sigma}_+, \hat{H}] = i\omega_0 \hat{\sigma}_+ - i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_z = i\omega_0 \hat{\sigma}_+ - i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* [s \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_z + (1-s) \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_z] \\ &= i\omega_0 \hat{\sigma}_+ - i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* [s \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_z + (1-s) \hat{\sigma}_z \hat{a}_{\vec{k}, \mu}^\dagger] \quad \text{since } [\hat{\sigma}_z(t), \hat{a}_{\vec{k}, \mu}^\dagger(t)] = 0 \end{aligned}$$

adding + subtracting $s g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_z$

$$\frac{d}{dt} \hat{\sigma}_z = -\frac{i}{\hbar} [\hat{\sigma}_z, \hat{H}] = -2i \sum_{\vec{k}, \mu} (g_{\vec{k}, \mu} \hat{\sigma}_+ \hat{a}_{\vec{k}, \mu} - g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^\dagger \hat{\sigma}_-) = -2i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \hat{\sigma}_+ \hat{a}_{\vec{k}, \mu} + \text{h.c.}$$

$$= -2i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} [s \hat{\sigma}_+ \hat{a}_{\vec{k}, \mu} + (1-s) \hat{a}_{\vec{k}, \mu} \hat{\sigma}_+] + \text{h.c.} \quad (\text{using same trick as in (b)})$$

Choose:

$0 \leq s \leq 1$: $s=0$ (Normal order), $s=1$ (anti-normal order), $s=\frac{1}{2}$ (Symmetric order)

(b) We solve formally:

$$\hat{a}_{\vec{k}, \mu}(t) = \underbrace{\hat{a}_{\vec{k}, \mu}(0) e^{-i\omega_k t}}_{\hat{a}_{\vec{k}, \mu}^{\text{free}}(t)} - i \underbrace{g_{\vec{k}, \mu}^* \int_0^t dt' e^{-i\omega_k(t-t')}}_{\delta \hat{a}_{\vec{k}, \mu}(t)} \hat{\sigma}_-(t')$$

$$\hat{\sigma}_+(t) = \underbrace{\hat{\sigma}_+(0) e^{i\omega_0 t}}_{\hat{\sigma}_+^{\text{free}}(t)} - i \underbrace{\int_0^t e^{i\omega_0(t-t')} \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^\dagger(t') \hat{\sigma}_z(t')}_{\delta \hat{\sigma}_+(t)}$$

$$\hat{\sigma}_z(t) = \underbrace{\hat{\sigma}_z(0)}_{\hat{\sigma}_z^{\text{free}}} - 2i \underbrace{\int_0^t dt' \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \hat{\sigma}_+(t') \hat{a}_{\vec{k}, \mu}(t')}_{\delta \hat{\sigma}_z(t)} + \text{h.c.}$$

Now we iterate, and keep terms only to first order in $g_{\vec{k}, \mu}$ (this is the first Born approximation)

$$\Rightarrow \delta \hat{a}_{\vec{k}, \mu}(t) = -i g_{\vec{k}, \mu}^* \int_0^t dt' e^{-i\omega_k(t-t')} \hat{\sigma}_-^{\text{free}}(t') = -i g_{\vec{k}, \mu}^* \int_0^t dt' e^{i(\omega_{cg} - \omega_k)(t-t')} \hat{\sigma}_-^{\text{free}}(t') \quad (\text{since } \hat{\sigma}_-^{\text{free}}(t) = e^{-i\omega_{cg}t} \hat{\sigma}_- |0\rangle)$$

$$\delta \hat{\sigma}_+^{\text{free}}(t) = -i \int_0^t e^{i\omega_{cg}(t-t')} \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* \hat{a}_{\vec{k}, \mu}^{\text{free}}(t') \hat{\sigma}_z^{\text{free}}(t') = -i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* \zeta(\omega_k - \omega_{cg}) \hat{a}_{\vec{k}, \mu}^{\text{free}}(t) \hat{\sigma}_z^{\text{free}}$$

$$\delta \hat{\sigma}_z^{\text{free}}(t) = -2i \int_0^t dt' \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \hat{\sigma}_+^{\text{free}}(t') \hat{a}_{\vec{k}, \mu}^{\text{free}}(t') + \text{h.c.} = -2i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \int_0^t e^{+i(\omega_{cg} - \omega_k)(t'-t)} \hat{\sigma}_+^{\text{free}}(t') \hat{a}_{\vec{k}, \mu}^{\text{free}}(t') + \text{h.c.}$$

$$\text{where } \zeta(\omega_{cg} - \omega_k) \equiv \int_0^t e^{i(\omega_{cg} - \omega_k)(t-t')} dt' \rightarrow \int_0^\infty e^{i(\omega_{cg} - \omega_k)\tau} d\tau = \pi \delta(\omega_{cg} - \omega_k) + i \mathcal{P} \left[\frac{1}{\omega_{cg} - \omega_k} \right]$$

In the Markoff approximation

(c) We take as the state (constant in the Heisenberg picture): $|\psi\rangle_{AF} = |\psi\rangle_A \otimes |0\rangle_F$

$$\text{Note: } \hat{a}_{\vec{k}, \mu}^{\text{free}} |0\rangle = 0, \quad \langle 0 | \hat{a}_{\vec{k}, \mu}^{\text{free}} |0\rangle = 0$$

Taking the expectation value of atomic Heisenberg equations of motion:

$$\frac{d}{dt} \langle \hat{\sigma}_+ \rangle = i\omega_{cg} \langle \hat{\sigma}_+ \rangle - i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* s \langle \psi_{AF} | (\hat{a}_{\vec{k}, \mu}^{\text{free}} + \delta \hat{a}_{\vec{k}, \mu}^{\text{free}}) (\hat{\sigma}_z^{\text{free}} + \delta \hat{\sigma}_z^{\text{free}}) | \psi_{AF} \rangle$$

$$- i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} (1-s) \langle \psi_{AF} | (\hat{\sigma}_z^{\text{free}} + \delta \hat{\sigma}_z^{\text{free}}) (\hat{a}_{\vec{k}, \mu}^{\text{free}} + \delta \hat{a}_{\vec{k}, \mu}^{\text{free}}) | \psi_{AF} \rangle$$

$$\Rightarrow \frac{d}{dt} \langle \hat{\sigma}_+ \rangle = i\omega_{cg} \langle \hat{\sigma}_+ \rangle - i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu}^* \left[s \langle \psi_{AF} | \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} \hat{\sigma}_z^{\text{free}} | \psi_{AF} \rangle + (1-s) (\langle \psi_{AF} | \hat{\sigma}_z^{\text{free}} \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle + \langle \psi_{AF} | \delta \hat{\sigma}_z^{\text{free}} \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle) \right]$$

$$\frac{d}{dt} \langle \hat{\sigma}_z \rangle = 2i \sum_{\vec{k}, \mu} (g_{\vec{k}, \mu} \left[s \langle \psi_{AF} | (\hat{\sigma}_+^{\text{free}} + \delta \hat{\sigma}_+^{\text{free}}) (\hat{a}_{\vec{k}, \mu}^{\text{free}} + \delta \hat{a}_{\vec{k}, \mu}^{\text{free}}) | \psi_{AF} \rangle + (1-s) \langle \psi_{AF} | (\hat{a}_{\vec{k}, \mu}^{\text{free}} + \delta \hat{a}_{\vec{k}, \mu}^{\text{free}}) (\hat{\sigma}_+^{\text{free}} + \delta \hat{\sigma}_+^{\text{free}}) | \psi_{AF} \rangle \right] + \text{h.c.})$$

$$\Rightarrow \frac{d}{dt} \langle \hat{\sigma}_z \rangle = 2i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \left[s \langle \psi_{AF} | \hat{\sigma}_+^{\text{free}} \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle + (1-s) (\langle \psi_{AF} | \hat{a}_{\vec{k}, \mu}^{\text{free}} \delta \hat{\sigma}_+^{\text{free}} | \psi_{AF} \rangle + \langle \psi_{AF} | \hat{\sigma}_+^{\text{free}} \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle) \right] + \text{h.c.}$$

(d) We now put it all together:

$$\langle \psi_{AF} | \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} \hat{\sigma}_z^{\text{free}} | \psi_{AF} \rangle = +i g_{\vec{k}, \mu}^* \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}}(t) \hat{\sigma}_z^{\text{free}}(t) \rangle \approx -i g_{\vec{k}, \mu}^* \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}}(t) \rangle$$

(to lowest order)

$$\langle \psi_{AF} | \hat{\sigma}_z^{\text{free}} \delta \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle = +i g_{\vec{k}, \mu} \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_z^{\text{free}} \hat{\sigma}_+^{\text{free}} \rangle = +i g_{\vec{k}, \mu} \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}}(t) \rangle$$

$$= \hat{\sigma}_+^{\text{free}}(t)$$

$$\langle \psi_{AF} | \delta \hat{\sigma}_z^{\text{free}} \hat{a}_{\vec{k}, \mu}^{\text{free}} | \psi_{AF} \rangle = -2i \sum_{\vec{k}, \mu} g_{\vec{k}, \mu} \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}} \rangle \langle 0 | \hat{a}_{\vec{k}, \mu}^{\text{free}} \hat{a}_{\vec{k}, \mu}^{\text{free}} | 0 \rangle = -2i g_{\vec{k}, \mu} \zeta^*(\omega_{cg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}}(t) \rangle$$

$\delta_{\vec{k}, \mu} \delta_{\vec{k}, \mu}$

$$\Rightarrow \left(\frac{d}{dt} - i\omega_{eg} \right) \langle \hat{\sigma}_+ \rangle = \left[s \left(- \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg} - \omega_k) \right) + (1-s) \left(+ \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg} - \omega_k) - 2 \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg} - \omega_k) \right) \right] \langle \hat{\sigma}_+ \rangle$$

$$\text{Aside: } \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg} - \omega_k) = \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \left(\pi \delta(\omega_{eg} - \omega_k) - i \mathcal{P} \left(\frac{1}{\omega_{eg} - \omega_k} \right) \right) = \frac{\Gamma}{2} + i\delta$$

$$\text{where } \Gamma = 2\pi \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \delta(\omega_k - \omega_{eg}), \quad \hbar\delta = \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \mathcal{P} \left[\frac{1}{E_g + \hbar\omega_k - E_e} \right]$$

$$\Rightarrow \left(\frac{d}{dt} - i\omega_{eg} \right) \langle \hat{\sigma}_+ \rangle = s \underbrace{\left(-\frac{\Gamma}{2} - i\delta \right)}_{\text{radiation reaction}} \langle \hat{\sigma}_+ \rangle + (1-s) \left(\underbrace{\left(\frac{\Gamma}{2} + i\delta \right)}_{\text{radiation reaction}} - 2 \underbrace{\left(\frac{\Gamma}{2} + i\delta \right)}_{\text{vacuum fluct}} \right) \langle \hat{\sigma}_+ \rangle$$

Where the vacuum terms arise from $\hat{a}_{\vec{k}\mu}^{\text{free}}(t)$ and the radiation reaction from $\delta\hat{a}(t)$

Note: In normal order ($s=1$), the entire level shift arises from radiation reaction.

In anti-normal order, ($s=0$), by vacuum fluctuation and radiation reaction contribute to δ .

In symmetric order ($s=\frac{1}{2}$), the entire level shift arises from vacuum fluctuations.

Now let's find the remaining correlation functions:

$$\begin{aligned} \langle \hat{\sigma}_+^{\text{free}} \delta\hat{a}_{\vec{k}\mu} \rangle &= -i g_{\vec{k}\mu}^* \zeta(\omega_{eg} - \omega_k) \langle \hat{\sigma}_+^{\text{free}} \hat{\sigma}_-^{\text{free}} \rangle = -i g_{\vec{k}\mu}^* \zeta(\omega_{eg} - \omega_k) \langle |e\rangle \langle e| \rangle \\ &= -i g_{\vec{k}\mu}^* \zeta(\omega_{eg} - \omega_k) \left\langle \frac{1 + \hat{\sigma}_z}{2} \right\rangle \end{aligned}$$

$$\langle \delta\hat{a}_{\vec{k}\mu} \hat{\sigma}_+^{\text{free}} \rangle = -i g_{\vec{k}\mu}^* \zeta(\omega_{eg} - \omega_k) \langle |g\rangle \langle g| \rangle = -i g_{\vec{k}\mu}^* \zeta(\omega_{eg} - \omega_k) \left\langle \frac{1 - \hat{\sigma}_z}{2} \right\rangle$$

$$\langle \hat{a}_{\vec{k}\mu}^{\text{free}} \delta\hat{\sigma}_+ \rangle = -i \sum_{\vec{k}', \mu'} g_{\vec{k}'\mu'}^* \zeta(\omega_k - \omega_{eg}) \langle 0 | \hat{a}_{\vec{k}\mu}^{\text{free}} \hat{a}_{\vec{k}'\mu'}^{\text{free}} | 0 \rangle \langle \hat{\sigma}_z \rangle = -i g_{\vec{k}\mu}^* \zeta(\omega_k - \omega_{eg}) \langle \hat{\sigma}_z \rangle$$

$$\Rightarrow \frac{d}{dt} \langle \hat{\sigma}_z \rangle = s \left[\sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \left[\zeta(\omega_{eg} - \omega_k) + \text{c.c.} \right] (1 + \langle \hat{\sigma}_z \rangle) \right] + (1-s) \left[-2 \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \left(\zeta(\omega_k - \omega_{eg}) + \text{c.c.} \right) \langle \hat{\sigma}_z \rangle - \sum_{\vec{k}, \mu} |g_{\vec{k}\mu}|^2 \left(\zeta(\omega_{eg} - \omega_k) + \text{c.c.} \right) (1 - \langle \hat{\sigma}_z \rangle) \right]$$

$$\Rightarrow \frac{d}{dt} \langle \hat{\sigma}_z \rangle = s \underbrace{\left(-\Gamma \langle \hat{\sigma}_z \rangle - \Gamma \right)}_{\text{radiation reaction}} + (1-s) \left[\underbrace{-2\Gamma \langle \hat{\sigma}_z \rangle}_{\text{vacuum fluct.}} + \underbrace{\left(\Gamma \langle \hat{\sigma}_z \rangle - \Gamma \right)}_{\text{radiation reaction}} \right]$$

- In normal order, entire decay rate attributed to radiation reaction

- In anti-normal order both contribute

- In symmetric order, the vacuum "stabilizes" the ground state